Harris Criteria for Quantum systems

Above we showed that for a classical system in d-dimensions, when $d\nu > 2$, disorder in the energy operator is perturbatively irrelevant.

One might naively think that for a quantum system at $T=0$, the criteria will get modified as $d \to d+z$. However, that is not the case because the impurities are fixed in space.

The heuristic argument we discussed

$\Delta T_c \ll |T-T_c|$ is still valid,

ΔT_c again goes like $\xi^{-d/2}$ and $|T-T_c|$ as $\xi^{-\frac{1}{\nu}}$ $\Rightarrow$ one still obtains $d\nu > 2$ for disorder to be irrelevant.

To see this using replica trick,

Schematically :

$$Z(m) = \int D\varphi \, e^{-\left[\int d^dx\, d\tau\, S_0(\varphi(x,\tau)) + \varphi^2(x,\tau)\, m(x)\right]}$$

Crucially, since the disorder is quenched, $m(x)$ is only a function of x and not (x,τ).

$$\Rightarrow \overline{Z^n(m)} = \int D\, m(x)\, Z^n(m)\, e^{-\int d^dx\, m^2(x)/2\Delta}$$

where we are assuming gaussian disorder for simplicity.

$$\Rightarrow \overline{Z^n(m)} = \int \prod D\varphi_a \, e^{-\sum_a \left[\int d^dx\, d\tau\, S_0(\varphi^a(x,\tau))\right]}$$

$$\int D\, m(x)\, e^{-\int d^dx\, d\tau \sum_a \varphi_a^2(x,\tau)\, m(x) - \int d^dx\, m^2(x)}$$

$$= \int \prod D\varphi_a \, e^{-\sum_a \int d^dx\, d\tau\, S_0(\varphi^a(x,\tau))} \, e^{\Delta \sum_{a,b} \int \varphi_a^2(x,\tau_1)\, \varphi_b^2(x,\tau_2)\, \underbrace{d\tau_1\, d\tau_2\, d^dx}_{}}$$

Note the double integ. over time and single integration over space.

Now all we need is the scaling dimension of the operator $\int d\tau_1\, d\tau_2\, d^d x\; \varphi^2_a(x,\tau_1)\, \varphi^2_b(x,\tau_2)$ at the clean fixed point.

If the dynamic exponent is z, scaling dimension of energy operator φ^2

$$= d + z - \frac{1}{\nu}.$$

Under RG, $x \to \frac{x}{b}$, $\tau \to \frac{\tau}{b^z}$

$\Rightarrow$ RG eigenvalue of $\int d\tau_1\, d\tau_2\, d^d x\; \varphi^2_a(x,\tau_1)\, \varphi^2_b(x,\tau_2)$

$$= 2z + d - 2\left(d + z - \frac{1}{\nu}\right)$$

$$= -d + \frac{2}{\nu}.$$

Importantly, z cancels out and one recovers the Harris criteria: when $d > \frac{2}{\nu}$, disorder is irrelevant.

Perturbatively accessible disordered fixed point

Above we argued that when $d\nu > 2$ i.e. exponent $\alpha = 2 - d\nu < 0$, energy disorder is perturbatively irrelevant. One may wonder if α is positive and small, perhaps one will flow to a new fixed point that is accessible perturbatively. For 3d Ising, $\alpha \approx 0.11$, therefore such an approach is applicable there. Further, $\alpha = 0$ in $d = 2$ (Onsager's solution). Therefore, it's a reasonable assumption that in $d = 2 + \varepsilon$, $\alpha = O(\varepsilon)$.

As discussed above, the effective Hamiltonian after replica trick is given by,

$$H_{eff} = \sum_a H_0^a - \Delta \sum_r \sum_{a \neq b} E_a(r)\, E_b(r)$$

where $\Delta = \overline{m^2} - \overline{m}^2$. One may now find the RG flow of Δ using OPE. Note the negative sign in front of the Δ term.

Above, we found that the leading term is proportional to $2 - d_\nu = \alpha$ i.e.

$$\frac{d\Delta}{dl} = \alpha \Delta + \ldots$$

We now calulate the ... using OPE. Since in $\sum_a H^0_a$, replicas are decoupled

$\varepsilon_a(r)\, \varepsilon_b(r) \propto \delta_{ab}$. More specifically,

$$\varepsilon_a \varepsilon_b \sim \delta_{ab} + c\, \varepsilon_a \delta_{ab},$$

where 'c' is a universal OPE coefficient.

As we earlier discussed, $c = 0$ for $d = 2$ Ising due to the Ising duality which maps $\varepsilon \to -\varepsilon$. Therefore, its reasonable to assume that in $d = 2 + \varepsilon$, $c = O(\varepsilon)$ at most.

To calulate the RG flow of Δ we need,

$$\sum_{a \neq b} \varepsilon_a \cdot \varepsilon_b \sum_{c \neq d} \varepsilon_c \cdot \varepsilon_d$$

$$= \left[4(n-2) + 2c^2 \right] \sum_{a \neq b} \varepsilon_a \varepsilon_b$$

$$\Rightarrow \quad \frac{d(-\Delta)}{dl} = \alpha(-\Delta) - [4(n-2)+2c^2](-\Delta)^2$$

$$\Rightarrow \quad \frac{d\Delta}{dl} = \alpha\Delta + [4(n-2)+2c^2]\Delta^2$$

Since $\alpha = O(\varepsilon)$, $c = O(\varepsilon)$, one may drop the c^2 term at $O(\varepsilon)$. Taking $n \to 0$ limit (!), one finds a stable fixed point at

$$\Delta^* = \frac{\alpha}{8} = O(\varepsilon).$$

To find the exponent ν, we need RG flow for t,

$$\frac{dt}{dl} = y_t^0\, t + 2C_{tOt}\, t\,\Delta$$

where $O = \sum_{a\neq b} \varepsilon_a \varepsilon_b$ and y_t^0 is the RG eigenvalue at the clean fixed point.

Thus, we need the OPE $\sum_{a\neq b} \varepsilon_a \varepsilon_b \sum_c \varepsilon_c$

Plugging in the above expression, one finds

$$\sum_{a\neq b} \varepsilon_a \varepsilon_b \sum_c \varepsilon_c = 2(n-1)\sum_c \varepsilon_c + \cdots$$

$$\Rightarrow \frac{dt}{dl} = y^0_t\, t + 4(n-1)\,\Delta t$$

$$y^0_t = \nu_{clean}^{-1} = \frac{d}{2-\alpha_{clean}} \approx \frac{2+\varepsilon}{2-\alpha}$$

$$\approx 1 + \frac{\varepsilon}{2} + \frac{\alpha}{2}.$$ Setting $n=0$,

$$\Rightarrow \quad y_t = y^0_t - 4\Delta^*$$

$$\approx 1 + \frac{\varepsilon}{2} + \frac{\alpha}{2} - \frac{\alpha}{2} \approx 1 + \frac{\varepsilon}{2}$$

$$\Rightarrow \quad \nu_{disorder} = y_t^{-1} \approx 1 - \frac{\varepsilon}{2}.$$

$$\alpha_{disorder} = 2 - d\nu_{disorder}$$

$$= 2 - (2+\varepsilon)(1 - \tfrac{\varepsilon}{2})$$

$$= 0 + O(\varepsilon^2)$$

There's a rigorous theorem due to Chages et al that at the disordered fixed point, $\alpha < 0$.

Rare region Effects in disordered systems.

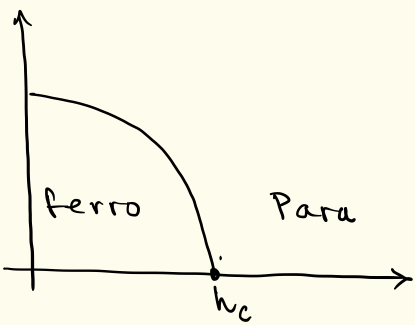
(~ Griffiths effects)

So far we have encountered power-law behavior at critical points. However, in disordered systems one can obtain power-law singularities even in phases that would have a gapped spectrum in the absence of disorder. Somewhat counterintuitively such singularities arise due to rare region effects.

Consider the transverse-field Ising model in d+1-dimensions; in the presence of disorder:

$$H = -J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - \sum_i h_i \sigma_i^x$$

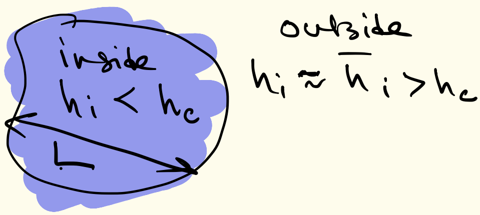
When $h_i = h \ \forall i$, the phase diagram is



At $T=0$, for $h > h_c$, the gap scales as $\Delta \sim |h-h_c|^{\nu}$

Therefore, all physical observables such as density of states vanish for energies less than the gap.

Now lets consider the effect of disorder. Due to disorder, there is non-zero probability of a region locally having h_i which is less than h_c of the clean system i.e. locally system wants to order. Lets estimate the probability that there exist a connected cluster of size L^d such that all spins within this cluster have $h_i < h_c$ while the average h_i, i.e. $\overline{h_i} > h_c$.

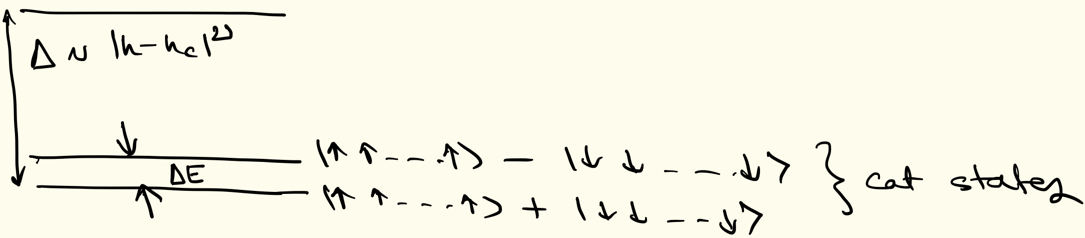


For example, if $P(h_i) = e^{-(h_i - \overline{h})^2/\sigma^2}$ then there is a small but non-zero probability p of obtaining $h_i = \overline{h} \pm \sigma^2$. Since we require that all spins inside this cluster have $h_i < h_c$

$\Rightarrow$ probability $P(L)$ for this to happen

$$= (p)^{L^d} \sim e^{-cL^d}$$

Within each such cluster, the system would tend to break symmetry spontaneously, except that due to finite size, the ground state degeneracy will get lifted due to the transverse field. The spectrum for a single such cluster would look like



The two cat states will mix at $O(L^d)$ in perturbation theory in h $\Rightarrow$

$$\Delta E \sim e^{-aL^d}$$

Altogether we have regions which are exponentially rare, $P(L) \sim e^{-cL^d}$, and within each such region, there are two low lying states with energy splitting $\Delta E \sim e^{-aL^d}$.

Now consider some probe which measures the density of low lying <u>excitations</u> in the system. One such probe is dynamic spin susceptibility $\mathrm{Im}\, \chi(\omega)$

$$\sim \sum_{n \neq 0} |\langle n | \sigma^z | 0 \rangle|^2 \, \delta(\omega - (E_n - E_0)).$$

If the system were clean, then the signal would be zero for $\omega < \Delta \sim |h - h_c|$

However, in the presence of rare regions, this result gets dramatically modified — The contribution from the rare region to any such probe would be,

$$\text{Signal}(\omega) \sim \int dL \; P(L) \; \delta(\omega - \Delta E)$$

$$\sim \int dL \; e^{-cL^d} \; \delta(\omega - e^{-aL^d})$$

Recall $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x)|_{x=x_i}}$ where

x_i are zeros of $f(x)$

$$\Rightarrow \quad \delta\left(\omega - e^{-aL^d}\right) = \frac{\delta(L - L_0)}{a\,d\,L_0^{d-1} e^{-aL_0^d}}$$

where $\omega = e^{-aL_0^d}$

$$\Rightarrow \quad \text{Signal}(\omega) \sim \frac{e^{-cL_0^d}}{e^{-aL_0^d}\, L_0^{d-1}}$$

$$\sim \frac{\omega^{c/a-1}}{\left[\log\left(\frac{1}{\omega}\right)\right]^{(d-1)/d}}$$

Therefore, one finds power-law behavior reminiscent of critical points without any actual critical phenomena. The power $\omega^{c/a-1}$ is non-universal, unlike actual critical phenomena.

Strong Disorder RG

There exist random systems where the disorder becomes stronger and stronger as one probes lower energy i.e. under RG flow disorder strength flows to infinity. Strong disorder RG provides a scheme to calculate physical properties of such systems.

One over-arching theme of such systems is that rare events/regions can dominate the physical properties, a bit like the example discussed above for random-field Ising model.

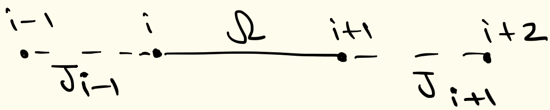
To illustrate the basic idea, consider the random bond Heisenberg model for spin-1/2 chain:

$$H = \sum_i J_i \vec{S}_i \cdot \vec{S}_{i+1}$$

Let's assume that the distribution of $\{J_i\}$ is rather wide with all $J_i < \Omega$.

Let's find the location of the strongest bond, which, by definition, has $J_i = \Omega$

In the vicinity of this bond, the system looks like



Due to the wide-distribution of $\{J_i\}$, it is very likely that $J_{i-1}, J_{i+1} \ll \Omega$. When $\frac{\Omega}{J_{i-1}}, \frac{\Omega}{J_{i+1}} = \infty$, then spins i and $i+1$ form a spin-singlet $(\Omega > 0)$, and therefore drop out from the system. Therefore, for a finite but large $\frac{\Omega}{J_{i-1}}, \frac{\Omega}{J_{i+1}}$, one may want to integrate out the spins i and $i+1$ to obtain an effective Hamiltonian for spins $i-1$ and $i+2$

$$H_{eff}(\text{spins } i-1, i+2) = \frac{\langle s | V | t \rangle \langle t | V | s \rangle}{E_t - E_s}$$

where $V = J_{i-1} \vec{S}_{i-1} \vec{S}_i + J_{i+1} \vec{S}_{i+1} \cdot \vec{S}_{i+2}$

$|s\rangle$ = singlet state for spins $i, i+1$

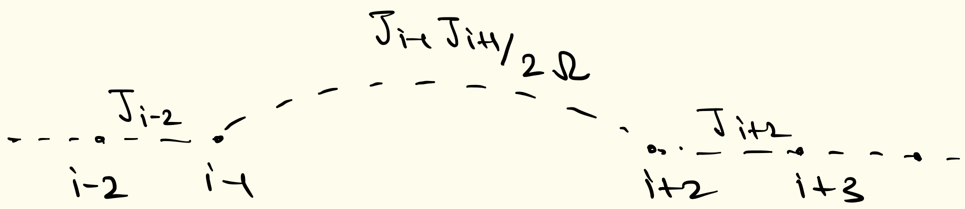
$|t\rangle$ = triplet state for spins $i, i+1$

Since $\epsilon_t - \epsilon_s \sim \Omega$, one may guess from the SU(2) symmetry that $H_{eff} \propto \frac{J_{i-1} J_{i+1}}{\Omega} \vec{S}_{i-1} \cdot \vec{S}_{i+2}$

Doing the actual calculation, one obtains

$$H_{eff} = \frac{J_{i-1} J_{i+1}}{2\Omega} \vec{S}_3 \cdot \vec{S}_4 .$$

Therefore after integrating out 1, 2 the new system looks like



Therefore, the probability distribution $P(\{J_i\})$ gets modified in two ways:

(i) We have removed a bond with strength Ω. This removal means that now the strongest bond will have a strength $< \Omega$. This effectively corresponds to changing the cut-off.

(ii) We have added a bond of strength

$$\frac{J_{i-1} J_{i+1}}{2\Omega}.$$

One just repeats this process over and over again, keeping track of the probability distribution $P(J, \Omega)$.

It is convenient to define new variables

$$x_i = \log\left(\frac{\Omega}{J_i}\right), \qquad y = \log\left(\frac{\Omega_0}{\Omega}\right)$$

where Ω is the instantaneous cutoff i.e the strongest bond at a given instant in RG. Note that $0 < x < \infty$. Ω_0 is arbitrary (say $\Omega_0 = 1$).

The result we earlier obtained where

$$J_{eff} = \frac{J_{i-1} J_{i+2}}{2\Omega}$$ in these variables

becomes $$\log\left(\frac{\Omega}{J_{eff}}\right) = \log\left(\frac{\Omega}{J_{i-1}}\right) + \log\left(\frac{\Omega}{J_{i+1}}\right) + \log(2)$$

or, $$x_{eff} = x_{i-1} + x_{i+1} + \log(2).$$

Let's derive the RG eqn:

(i) When we remove bonds between Ω and $\Omega - d\Omega$ (since they are the strongest at a given instant),

$$x_i = \log\left(\frac{\Omega}{J_i}\right) \to \log\left(\frac{\Omega - d\Omega}{J_i}\right)$$

$$= x_i - \frac{d\Omega}{\Omega} = x_i - dy$$

$$\Rightarrow \delta x_i = -dy.$$

$\Rightarrow$ $dP(x, y)$ due to this procedure

$$= \frac{\partial P}{\partial x} \delta x = -\frac{\partial P}{\partial x} dy$$

(ii) When we add a bond of strength $x_{eff} = x_{i-1} + x_{i+1} + \log(2)$, the corresponding probability change is proportional to

$$\delta P(x) \propto \int dx_1 dx_2 \left[P(x_1)\, P(x_2)\, \delta\left(x_{eff} - (x_1 + x_2 + \log(2))\right)\right]$$

The proportionality constant is the fractional of spins that are the strongest bond in the range $(0, dy)$

i.e. $P(0)\, dy$.

$$\Rightarrow \quad \delta P(x) = P(0)\, dy \int dx_1 dx_2\, P(x_1)\, P(x_2)\, \delta(x - (x_1 + x_2))$$

where we have dropped $\log(2)$ since a typical $x \gg \log(2)$. Combining the two terms:

$$dP = -\frac{\partial P}{\partial x} dy + P(0)\, dy \int dx_1 dx_2\, P(x_1)\, P(x_2)\, \delta(x - (x_1 + x_2))$$

$$\Rightarrow \quad \frac{dP}{dy} = \frac{\partial P}{\partial x} + P(0)\int_0^\infty dx_1 \int_0^\infty dx_2 \; P(x_1)\, P(x_2)\, \delta(x-(x_1+x_2))$$

This is the RG equation one needs to solve. An ansatz that solves this is

$$P(x,y) = f(y)\, e^{-f(y)\,x}$$

Plugging this into the RG equation, one finds $\frac{\partial f}{\partial y} = -f^2 \Rightarrow f(y) = \frac{1}{y}$

$$\Rightarrow \quad \boxed{P(x,y) = \frac{e^{-x/y}}{y}}$$

Now recall that
$$x = \log(\Omega/J)$$
$$y = \log(\Omega_0/\Omega)$$

$$\Rightarrow \quad P_\Omega(J) \sim \frac{1}{y}\left(\frac{\Omega}{J}\right)^{1-\frac{1}{y}}$$

Therefore, one obtains a rather wide distribution, which justifies the whole scheme.

Physical consequences:

Lets compute the number of surviving spins (i.e. those that have not been integrated out) as a f^n of Ω, the cutoff.

Each RG step removes two spins, both of which are at $J = \Omega$ i.e. $x = 0$

$\Rightarrow$ Change in number of spins as $\Omega \to \Omega - d\Omega$

$$dN = -2\, P(x=0)\, N\, dy$$

From the above solution $P(x=0) = \frac{1}{y}$

$$\Rightarrow \quad \frac{dN}{dN} = -2\frac{dy}{y} \quad \Rightarrow \quad N(y) \sim \frac{N_0}{y^2}$$

Average distance L between spins $= \frac{N_0}{N(y)} \sim y^2$

$$\Rightarrow \quad L(\Omega) \sim y^2 \sim \log^2\left(\frac{\Omega_0}{\Omega}\right)$$

$\Rightarrow$ Excitation energy $\sim \Omega \sim e^{-1/\sqrt{L}}$.

Contrast this with a typical non-disordered system where Energy $\sim \dfrac{1}{(\text{length})^{z}}$.

This also allows us to estimate spin susceptibility at temperature T.

$$\chi(T) = [\text{fraction of spins surviving at energy scale } T] \times \underbrace{\text{Curie susceptibility}}_{\sim 1/T}$$

$$\sim \frac{1}{\log^2\left(\frac{\Omega_0}{T}\right)} \times \frac{1}{T}$$

$$\sim \frac{1}{T \log^2(\Omega_0/T)}$$